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# On the Properties of the Weibull-Burr III Distribution and its Application to Uncensored and Censored Survival Data

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# On the Properties of the Weibull-Burr III Distribution and its Application to Uncensored and Censored Survival Data

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*Twelve different families of cumulative distributions that are used to model real life data were introduced by Burr (1942). Burr III distribution is among these families of cumulative distributions. In this work, a four-parameter distribution is introduced to model real life scenarios called Weibull-Burr III distribution. The limiting behavior of the proposed distribution, hazard function, moments, skewness, kurtosis and quantile function is investigated; order statistics and entropy are also derived. The method of Maximum Likelihood Estimation technique was used in estimating the parameters of the proposed distribution. To prove the flexibility and performance of the distribution and Weibull-G family of distributions, censored and uncensored data sets are applied. The results suggest that the new compound distribution fit the real data and perform much better than its competitors for both censored and uncensored data.*

**Keywords:** Rényi Entropy, Weibull-Burr type III Distribution, Weibull-G Family

**JEL Classification**: C02, C22, I10

## 1.0 Introduction

Lifetime data can be modelled using several existing distributions. However, some of these lifetime data do not follow these existing distributions or are inappropriately described by them. Hence, the need to develop distributions that could better describes some of these phenomena and provide greater flexibility in the modelling of lifetime data than the baseline distributions. Therefore, many distributions have been developed and studied by many researchers. These include: An extended Lomax distribution by Lemonte and Cordeiro (2011), Beta-Burr X by Merovci *et al*. (2016), Beta-Normal by Eugene *et al.* (2002), Beta-Nakagami by Olanrewaju and Kazeem (2013), Kumaraswamy-Burr III by Behairy *et al.* (2016), Kumaraswamy-Paretor by Bourguignon *et al.* (2013), new Weibull-Paretor by Nasiru and

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Luguterah (2015), Transmuted Lomax Distribution by Ashour and Eltehiwy (2013), Transmuted new generalized Weibull by Khan *et al.* (2016), Weibull-exponential by Oguntunde *et al.* (2015), Weibull-Paretor by Alzaatreh *et al.* (2013), Weibull-Rayleigh by Merovci and Elbatal (2015), and many more.

In the last few years, different classes of the Weibull Generalized family of distributions have been proposed and studied by several researchers. These include: Alzaatreh *et al.* (2013), Bourguignon *et al*. (2014) and Nasiru and Luguterah (2015). The objective of this paper is to introduce a new compound distribution called the Weibull-Burr III distribution, study some statistical properties of the new distribution comprehensively, use MLE method to estimate the parameters of the proposed distribution and finally to use censored and uncensored survival data sets in fitting the new distribution and some of the existing Weibull-G family of distributions so as to compare the performances of the Weibull-G family of distributions.

## 2.0 Literature Review

### 2.1 Empirical literature

Alzaatreh *et al.* (2013) proposed a distribution called Weibull-Paretor, as a special case of the Weibull-G family. Statistical and mathematical properties of this distribution studied include; moments, moment generating function, hazard function and Shannon entropy. It was shown that the distribution is unimodal and the shape of the distribution can either be positively or negatively skewed. Modified Maximum Likelihood Estimation was used for the estimation of the parameters of the distribution.

Bourguignon *et al.* (2013) extended the two-parameter Paretor distribution by introducing two shape parameters. This is done by taking the baseline cumulative distribution of the generalized family of Kumaraswamy distribution to be the cumulative distribution of Paretor distribution. Detailed mathematical properties of the distribution were provided and method of Maximum Likelihood was applied in estimating the parameters of the distribution. It was further shown that the Kumaraswamy-Paretor distribution is superior to its sub-models when real data set is used in fitting these distributions.

Olanrewaju and Kazeem (2013) developed the Beta-Nakagami distribution using the link function of the Beta generalized distribution. Statistical properties of the Beta-Nakagami distribution such as the asymptotic behavior, moments, moment generating function among others were investigated. The parameters of this distribution were estimated using the Maximum Likelihood Method. Real data was used to fit the Beta-Nakagami distribution and Nakagami distribution. It was found that Beta-Nakagami distribution apart from being more flexible has better representation of data than Nakagami distribution.

A three-parameter distribution referred to as Weibull-Rayleigh distribution was studied by Merovci and Elbatal (2015). This distribution was proposed using the logit of Weibull-G proposed by Bourguignon *et al*. (2014). Like other researchers, different mathematical and statistical properties of the distribution were provided. The parameters of the distribution were estimated using both the least square estimation method and MLE. Comparison of the distribution with other distributions such as Beta-Weibull, Exponentiated-Weibull and Weibull distributions reveal that the Weibull-Rayleigh distribution is a strong competitor for fitting real life data.

Another three-parameter distribution called Weibull-exponential was proposed by Oguntunde *et al.* (2015) using the link function of the Weibull generalized family. Explicit expressions for some basic mathematical properties like moments, moment generating function, reliability, limiting behavior and order statistics of the distribution were derived. Like most researchers, MLE was used in estimating the parameters of the distribution. It was shown that Weibull-exponential is more useful as a life testing model than the exponential distribution.

Nasiru and Luguterah (2015) proposed a new distribution called new Weibull-Paretor distribution. The distribution was called new Weibull-Paretor because Alzaatreh *et al.* (2013) had already defined Weibull-Paretor distribution using the generator proposed by Alzaatreh *et al.* (2013). Nasiru and Luguterah (2015) studied various properties of the new Weibull-Paretor distribution and used MLE to estimate the parameters of the distribution. Application of real data set to the new Weibull-Paretor distribution revealed that the distribution provides a better fit in modelling real life data.

Behairy *et al.* (2016) extended the Burr III distribution using the logit of Kumaraswamy-G family. Explicit expressions for the moments, density functions of the order statistics, Rényi entropy, quantiles and moment generating function were provided. MLE method was applied under Type II censored sample to estimate the parameters of the model and Monte Carlo simulation was performed to investigate the precision of the estimates.

Beta-Burr X distribution was developed and studied by Merovci *et al.* (2016). Comprhensive mathematical properties of this distribution were provided. Also, asymptotic confidence intervals for the parameters of the Beta-Burr X distribution were derived from the Fisher Information Matrix. Furthermore, simulation study was conducted to assess the performance of the model. Model fit of the distribution indicates that Beta-Burr X serves as a good alternative model for modelling positive real data in many areas.

## 2.2 Theoretical Framework (Weibull-G family of Distributions)

Let $X$ be a random variable from the Weibull distribution with parameters α and β, then the cumulative distribution function (cdf) and probability density function (pdf) of the Weibull generalized family of distribution (Weibull-G) due to Alzaatreh *et al*. is given by:

$$F(x)=\int_0^{-\log(1-G(x))}\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\,dx \tag{1}$$

and

$$f(x)=\alpha\beta\frac{g(x)}{\overline{G}(x)}\left[-\beta\log\left(\overline{G}(x)\right)\right]^{\alpha-1}\exp\left(-\left(-\beta\log\left(\overline{G}(x)\right)\right)^{\alpha}\right) \tag{2}$$

for $x > 0,\ \alpha,\beta\ > 0$

where $\overline{G}(x)=1-G(x)$; and $g(x)$ and $G(x)$ are the pdf and cdf of any baseline distribution and in our case the Burr type III distribution.

Another generator of the Weibull generalized family of distributions is the one proposed by Bourguignon *et al.* (2014). The cdf and pdf of the Weibull generalized family of distribution due to Bourguignon *et al.* (2014) are defined by:

$$F(x)=\int_0^{\frac{G(x)}{\overline{G}(x)}}\alpha\beta x^{\beta-1}\exp\left(-\alpha x^{\beta}\right)dx \tag{3}$$

$$f(x)=\alpha\beta g(x)\frac{\left(G(x)\right)^{\beta-1}}{\left(\overline{G}(x)\right)^{\beta+1}}\exp\left(-\alpha\left[\frac{G(x)}{\overline{G}(x)}\right]^{\beta}\right) \tag{4}$$

where $x > 0, \alpha > 0, \beta > 0$

A year after this generator was proposed, another form of Weibull generalized family of distribution was proposed by Nasiru and Luguterah (2015). The cdf and pdf due to Nasiru and Luguterah (2015) are given by:

$$F(x)=\int_0^{\frac{1}{1-G(x)}}\alpha\beta x^{\beta-1}\exp\left(-\alpha x^{\beta}\right)dx \tag{5}$$

$$f(x)=\frac{\alpha\beta g(x)}{\left(\overline{G}(x)\right)^{\beta+1}}\exp\left(-\alpha\left[\frac{1}{\overline{G}(x)}\right]^{\beta}\right) \tag{6}$$

where $x > 0, \alpha > 0, \beta > 0$

In this paper, we intend to develop a four-parameter model called Weibull-Burr III distribution using the Weibull generator proposed by Bourguignon *et al.* (2014). This generator has been used by other researchers to develop compound distributions such as: Weibull-Rayleigh distribution by Merovci and Elbatal (2015), Weibull-exponential distribution by Oguntunde *et al.* (2015) and so on.

## 3.0 Methodology

### 3.1 The Proposed Distribution

Our baseline distribution, the Burr III distribution with parameters $(\lambda,\ \gamma)$ has its cdf and pdf given by:

$$G(x;\ \lambda,\gamma)=\left(1+x^{-\lambda}\right)^{-\gamma} \tag{7}$$

and

$$g(x;\lambda,\gamma)=\lambda\gamma x^{-(\lambda+1)}\left(1+x^{-\lambda}\right)^{-(\gamma+1)} \tag{8}$$

respectively. Also, $\lambda > 0$ and $\gamma > 0$ are shape parameters.

Using the generator proposed by Bourguignon *et al.* (2014) the cdf of the proposed Weibull-Burr III distribution is given by:

$$F(x;\alpha,\beta,\lambda,\gamma)=1-\exp\left\{-\alpha\left[\left(1+x^{-\lambda}\right)^{\gamma}-1\right]^{-\beta}\right\} \quad (9)$$

and its corresponding pdf is given by:

$$f(x;\alpha,\beta,\lambda,\gamma)=\alpha\beta\lambda\gamma x^{-(\lambda+1)}\frac{\left(1+x^{-\lambda}\right)^{-(\gamma\beta+1)}}{\left(1-\left(1+x^{-\lambda}\right)^{-\gamma}\right)^{\beta+1}}\exp\left\{-\alpha\left[\left(1+x^{-\lambda}\right)^{\gamma}-1\right]^{-\beta}\right\} \quad (10)$$

where $x>0,\ \alpha>0,\ \beta>0,\ \lambda>0\ and\ \gamma>0$. In this distribution, $\beta$ is the scale parameter while $\alpha$, $\lambda$ and $\gamma$ are the shape parameters.

### 3.2 Investigation of the Proposed Distribution for a Proper PDF

A pdf is said to be proper if

$\int_{-\infty}^{\infty} f(x)dx = 1.$

To show that the proposed distribution is a proper pdf, we proceed as follows:

$$\int_0^{\infty} f(x)dx=\int_0^{\infty}\alpha\beta\lambda\gamma x^{-(\lambda+1)}\frac{\left(1+x^{-\lambda}\right)^{-(\gamma\beta+1)}}{\left(1-\left(1+x^{-\lambda}\right)^{-\gamma}\right)^{\beta+1}}\exp\left\{-\alpha\left[\left(1+x^{-\lambda}\right)^{\gamma}-1\right]^{-\beta}\right\}dx$$

$$\text{let}\, u=\left(1+x^{-\lambda}\right)^{\gamma}-1 \quad \Rightarrow dx=-\frac{(u+1)^{-\frac{1}{\gamma}-1}}{\lambda\gamma}\left[(u+1)^{\frac{1}{\gamma}}-1\right]^{-\frac{1}{\lambda}}du$$

Then $\int_0^{\infty} f(x)dx$ becomes

$$\int_0^{\infty} f(x)dx=\alpha\beta\int_0^{\infty} u^{-\beta-1}\exp(-\alpha u^{-\beta})du$$

Following the same procedure by letting $m=\alpha u^{-\beta}$ , the integral becomes:

$$\int_0^\infty f(x)\,dx = \alpha\beta\int_0^\infty \left(\frac{m}{\alpha}\right)^{\frac{\beta+1}{\beta}} \exp(-m)\frac{1}{\alpha\beta}\left(\frac{m}{\alpha}\right)^{-\frac{\beta+1}{\beta}} dm$$

$$= \int_0^\infty \exp(-m)\,dm$$

$$\int_0^\infty f(x)\,dx = \Gamma(1)$$

$$= 1$$

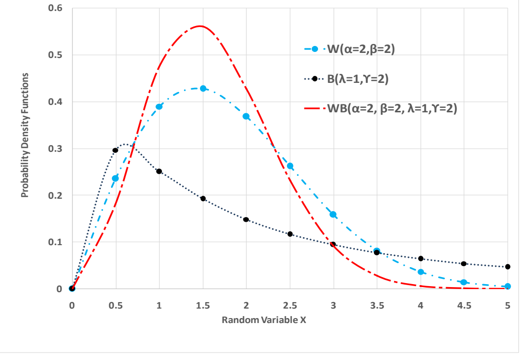


Figure 1a: Graph of the Three Distributions Weibull, Burr III and Weibull-Burr III (α, λ and γ = shape and β= scale parameters)

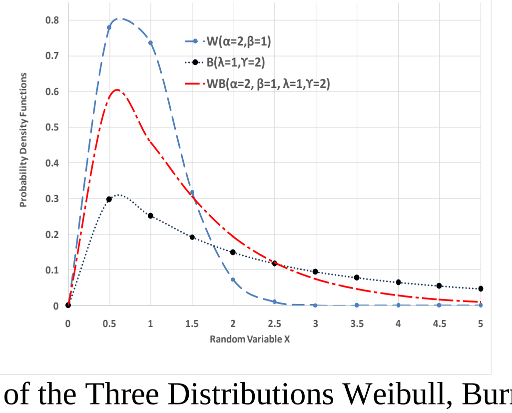


Figure 1b: Graph of the Three Distributions Weibull, Burr III and Weibull-Burr III (α, λ and γ = shape and β= scale parameters)

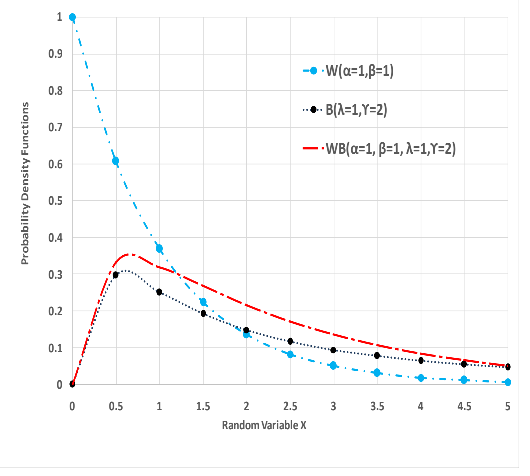


Figure 1c: Graph of the Three Distributions Weibull, Burr III and Weibull-Burr III ($\alpha$, $\lambda$ and $\gamma$ = shape and $\beta$= scale parameters)

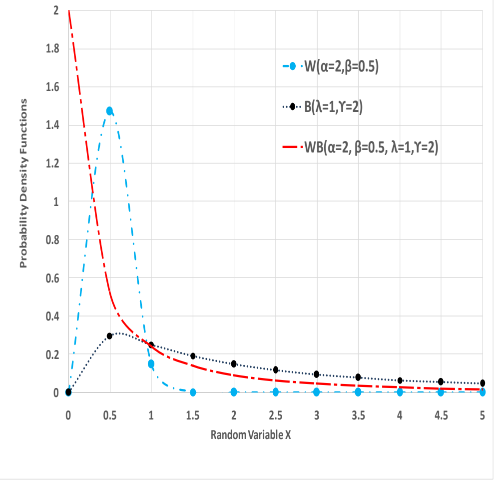


Figure 1d: Graph of the Three Distributions Weibull, Burr III and Weibull-Burr III ($\alpha$, $\lambda$ and $\gamma$ = shape and $\beta$= scale parameters)

Hence, the Weibull-Burr III distribution is a proper pdf. The graph of the pdf of Weibull-Burr III, Weibull and Burr III with different parameter values are given in Fig 1a – 1d. The proposed distribution appears to be much more flexible than the two parent distributions as can be glanced by varying the shape parameter values.

### 3.3 Asymptotic Behavior

We now investigate the asymptotic behavior of the Weibull-Burr III distribution as $x$ tends to zero and as $x$ tends to infinity.

$$\lim_{x\to 0}\alpha\beta\lambda\gamma x^{-(\lambda+1)}\frac{\left(1+x^{-\lambda}\right)^{-(\gamma\beta+1)}}{\left(1-\left(1+x^{-\lambda}\right)^{-\gamma}\right)^{\beta+1}}\exp\left\{-\alpha\left[\left(1+x^{-\lambda}\right)^{\gamma}-1\right]^{-\beta}\right\}=0$$

$$\text{since}\lim_{x\to 0}x^{-(\lambda+1)}\left(1+x^{-\lambda}\right)^{-(\gamma\beta+1)}=\lim_{x\to 0}x^{\lambda\gamma\beta-1}\left(1+x^{\lambda}\right)^{-(\gamma\beta+1)}=0$$

Applying the same technique, it can easily be shown that $\lim_{x\to\infty} f(x)=0$. It has been shown in the literature that, if $\lim_{x\to 0} f(x)=0$, and $\lim_{x\to\infty} f(x)=0$, then $f(x)$ has at least one mode (see Olanrewaju and Kazeem (2013) for more details). Hence, the Weibull-Burr III distribution has a mode.

### 3.4 Hazard Function

The hazard function which has an important application in survival (reliability) analysis is defined by:

$$h(x)=\frac{g(x)}{1-G(x)} \tag{11}$$

$$h(x)=\frac{\alpha\beta\lambda\gamma x^{-(\lambda+1)}\dfrac{\left(1+x^{-\lambda}\right)^{-(\gamma\beta+1)}}{\left(1-\left(1+x^{-\lambda}\right)^{-\gamma}\right)^{\beta+1}}\exp\left\{-\alpha\left[\left(1+x^{-\lambda}\right)^{\gamma}-1\right]^{-\beta}\right\}}{\exp\left\{-\alpha\left[\left(1+x^{-\lambda}\right)^{\gamma}-1\right]^{-\beta}\right\}}$$

Hence, the hazard function for Weibull-Burr III is given by:

$$h(x)=\alpha\beta\lambda\gamma x^{-(\lambda+1)}\frac{\left(1+x^{-\lambda}\right)^{-(\gamma\beta+1)}}{\left(1-\left(1+x^{-\lambda}\right)^{-\gamma}\right)^{\beta+1}} \tag{12}$$

The hazard function is the probability of failure in an infinitesimally small time period between $x$ and $x+\partial x$ given that the subject has survived up to time $x$. The graph of the hazard function of Weibull-Burr III is shown in Fig 2a and 2b.

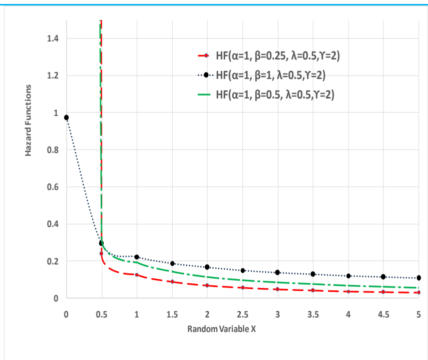


Figure 2a: Hazard Function for the Weibull- Weibull-Burr III (α, λ and γ = shape and β= scale parameters)

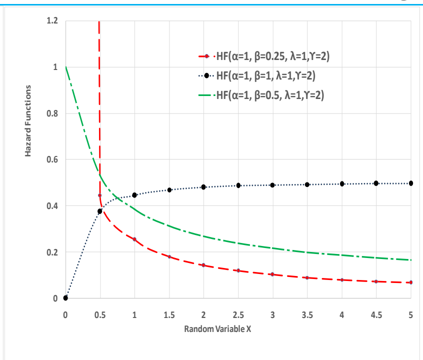


Figure 2b: Hazard Function for the Burr III (α, λ and γ = shape and β= scale parameters)

## 3.5 Moments, Skewness and Kurtosis

Moments can be used in studying some important properties such as dispersion, skewness and kurtosis of a distribution. Let $X$be a random variable from the Weibull-Burr III distribution, then the r$^{th}$ moment of $X$ is defined by:

$$E\left(X^r\right)=\int_0^\infty x^r.\alpha\beta\lambda\gamma x^{-(\lambda+1)}\frac{\left(1+x^{-\lambda}\right)^{-(\gamma\beta+1)}}{\left(1-\left(1+x^{-\lambda}\right)^{-\gamma}\right)^{\beta+1}}\exp\left\{-\alpha\left[\left(1+x^{-\lambda}\right)^{\gamma}-1\right]^{-\beta}\right\}dx \qquad (13)$$

Using Power Series expansion, $\exp\left\{-\alpha\left[\left(1+x^{-\lambda}\right)^{\gamma}-1\right]^{-\beta}\right\}$ can be written in the form:

$$\exp\left\{-\alpha\left[\left(1+x^{-\lambda}\right)^{\gamma}-1\right]^{-\beta}\right\}=\sum_{j=0}^{\infty}\frac{(-1)^j}{\Gamma(j+1)}\alpha^j\left[\left(1+x^{-\lambda}\right)^{\gamma}-1\right]^{-\beta j} \qquad (14)$$

where $\Gamma(.)$ is the gamma function.

Substituting (14) in (13) gives:

$$E\left(X^r\right)=\alpha\beta\lambda\gamma\int_0^\infty x^{r-\lambda-1}\sum_{j=0}^{\infty}\frac{(-1)^j}{\Gamma(j+1)}\alpha^j\left(1+x^{-\lambda}\right)^{-(\gamma\beta(j+1)+1)}\left[1-\left(1+x^{-\lambda}\right)^{-\gamma}\right]^{-(\beta(j+1)+1)}dx \qquad (15)$$

Since the limiting values of $\left(1+x^{-\lambda}\right)^{-\gamma}$ is between zero and one, as $x$ tends to zero and infinity respectively, then using Binomial Series expansion equation (15) becomes:

$$E\left(X^r\right)=\alpha\beta\lambda\gamma\sum_{j,k=0}^{\infty}\frac{(-1)^{j+k}\alpha^j\Gamma\left(\beta(j+1)+2\right)}{\Gamma(j+1)\Gamma(k+1)\Gamma\left(\beta(j+1)-k+2\right)}\int_0^{\infty}x^{r-\lambda-1}\left(1+x^{-\lambda}\right)^{-\gamma k-\left(\gamma\beta(j+1)+1\right)}dx \qquad (16)$$

taking $\int_0^{\infty}x^{r-\lambda-1}\left(1+x^{-\lambda}\right)^{-\gamma k-\left(\gamma\beta(j+1)+1\right)}dx$,

let $z=\gamma k+\left(\gamma\beta(j+1)+1\right)$, $m=\left(1+x^{-\lambda}\right)^{-z}$, then

$$dx=\frac{m^{-\frac{1}{z}-1}}{z\lambda}\left(m^{-\frac{1}{z}}-1\right)^{-\frac{1}{\lambda}-1}dm$$

Hence,

$$\int_0^{\infty}x^{r-\lambda-1}\left(1+x^{-\lambda}\right)^{-\gamma k-\left(\gamma\beta(j+1)+1\right)}dx=\frac{1}{z\lambda}\int_0^{1}m^{-\frac{1}{z}-\frac{1}{\lambda z}+\frac{r}{\lambda z}+\frac{1}{z^2}}\left(1-m^{\frac{1}{z}}\right)^{\frac{1}{\lambda}-\frac{r}{\lambda}-\frac{1}{z}}dm \qquad (17)$$

Letting $u=m^{\frac{1}{z}}$, the integral in (17) becomes:

$$\frac{1}{\lambda}B\left(\frac{r}{\lambda}+\frac{1}{z}-\frac{1}{\lambda}+z-1,\frac{1}{\lambda}-\frac{r}{\lambda}-\frac{1}{z}+1\right) \qquad (18)$$

where $B(.,.)$ is the beta function.

Substituting (18) in (16), we obtain the r$^{th}$ moment of Weibull-Burr III distribution as:

$$E\left(X^r\right)=\beta\gamma\sum_{j,k=0}^{\infty}\frac{(-1)^{j+k}\alpha^{j+1}\Gamma\left(\beta(j+1)+2\right)}{\Gamma(j+1)\Gamma(k+1)\Gamma\left(\beta(j+1)-k+2\right)}B\left(\frac{r}{\lambda}+\frac{1}{z}-\frac{1}{\lambda}+z-1,\frac{1}{\lambda}-\frac{r}{\lambda}-\frac{1}{z}+1\right)$$

The p$^{th}$ central moment can easily be obtained from moment about the origin:

$$\mu_p=\sum_{l=1}^{p}(-1)^l\,{}^pC_l\left(\mu_1'\right)^l\mu_{p-l}' \qquad (19)$$

while the p$^{th}$ cumulant of $X$ is given by:

$$\kappa_p = \mu_p ' - \sum_{l=0}^{p-1} {}^{p-1}C_{l-1} \kappa_l \mu_{p-l} ' \qquad (20)$$

From equation (20),

$$\kappa_2 = \mu_2 ' - (\mu_1 ')^2$$

$$\kappa_3 = \mu_3 ' - 3\mu_2 ' \mu_1 ' + 2(\mu_1 ')^3$$

$$\kappa_4 = \mu_4 ' - 4\mu_3 ' \mu_1 ' + 12\mu_2 '(\mu_1 ')^2 - 3(\mu_2 ')^2 - 6(\mu_1 ')^4$$

With these, the coefficients of skewness and kurtosis can easily be obtained as:

$$\text{Coefficient of skewness} = \frac{\kappa_3}{(\kappa_2)^{3/2}} \qquad (21)$$

$$\text{Coefficient of kurtosis} = \frac{\kappa_4}{\kappa_2^{\,2}} \qquad (22)$$

### 3.6 Quantile Function

The quantile distribution is used in the generation of random realizations from a given distribution. The quantile function of Weibull-Burr III distribution is given by:

$$Q(u) = \left[ \left\{ 1 + \left( \ell n (1-u)^{-\frac{1}{\alpha}} \right)^{-\frac{1}{\beta}} \right\}^{\frac{1}{\gamma}} - 1 \right]^{-\frac{1}{\lambda}} \qquad (23)$$

where $u$ is a random number generated from uniform distribution with parameters $0$ $and$ $1$

The median of Weibull-Burr III distribution can be obtained by substituting $u = \frac{1}{2}$ in equation (23) which gives:

$$\text{Median} = \left[ \left\{ 1 + \left( \frac{1}{\alpha} \ell n 2 \right)^{-\frac{1}{\beta}} \right\}^{\frac{1}{\gamma}} - 1 \right]^{-\frac{1}{\lambda}} \quad (24)$$

## 3.7 Order Statistics

Let $x_1, x_2, ..., x_n$ be independent random sample from a cumulative distribution function, $F(x)$, with an associated probability density function, $f(x)$. Then, the probability density function of the $i^{th}$ order statistics, $x_{(i)}$, is defined by:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)\left(F(x)\right)^{i-1}\left[1-F(x)\right]^{n-i} \quad (25)$$

Recall that the Binomial Series expansion of $(1-x)^n$ is given by:

$$(1-x)^n = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!k!} x^k$$

Using this expansion, equation (25) can be written as

$$f_{i:n}(x) = \sum_{k=0}^{n-i} \frac{(-1)^k n!}{(i-1)!(n-i-k)!k!} f(x)\left[F(x)\right]^{k+i-1} \quad (26)$$

and

$$\left[F(x)\right]^{k+i-1} = \sum_{p=0}^{i+k-1} (-1)^p \frac{(i+k-1)!}{(i+k-p-1)!\,p!} \exp\left\{-\alpha p\left[\left(1+x^{-\lambda}\right)^{\gamma}-1\right]^{-\beta}\right\} \quad (27)$$

Substituting (10) and (27) in (26) yields the $i^{th}$ order statistics for the Weibull-Burr III distribution given by:

$$f_{i:n}(x) = \sum_{k=0}^{n-i} \sum_{p=0}^{i+k-1} \delta_i f_i(x) \quad (28)$$

where $\delta_i = \frac{(-1)^{p+k} n!(i+k-1)!}{(i-1)!(n-i-k)!(i+k-p-1)!\,p!k!(p+1)}$ ;

$f_i(x)$ is the pdf of Weibull-Burr III with parameters $(\alpha(p+1),\beta,\lambda,\gamma)$. Using equation (28), several mathematical properties of Weibull-Burr III order statistics such as moments, ordinary moments, moment generating function, factorial moments and so on can be determined. For instance, the $S^{th}$ moment of $X_{i:n}$ can easily be obtain from the expression in (19) which is the moment of Weibull-Burr III with new parameters $(\alpha(p+1),\beta,\lambda,\gamma)$.

### 3.8 Rényi Entropy

Numerous entropies such as Rényi entropy, Shannon entropy, etc. have been developed and used in various disciplines and contexts. The entropy of a random variable, $X$, denoted by $I_R(r)$ is defined as a measure of the uncertainty about the outcome of a random experiment. Let $X$ be a random variable with pdf, $f(x)$, then the Rényi entropy is defined by:

$$I_R(r)=\frac{1}{1-r}\ell n\left[\int_{\square}\left(f(x)\right)^r dx\right]$$

for $r>0$ and $r\neq 1$

$$\int_0^\infty \left(f(x)\right)^r dx=\int_0^\infty (\alpha\beta\lambda\gamma)^r x^{-r(\lambda+1)}\frac{\left(1+x^{-\lambda}\right)^{-r(\gamma\beta+1)}}{\left(1-\left(1+x^{-\lambda}\right)^{-\gamma}\right)^{r(\beta+1)}}\exp\left\{-\alpha r\left[\left(1+x^{-\lambda}\right)^{\gamma}-1\right]^{-\beta}\right\}dx \qquad (29)$$

Using Power Series expansion, $\exp\left\{-\alpha r\left[\left(1+x^{-\lambda}\right)^{\gamma}-1\right]^{-\beta}\right\}$ can be written in the form:

$$\sum_{j=0}^{\infty}\frac{(-1)^j r^j\alpha^j}{j!}\left[\left(1+x^{-\lambda}\right)^{\gamma}-1\right]^{-\beta j} \qquad (30)$$

and substituting (30) in (29) yields:

$$(\alpha\beta\lambda\gamma)^r\sum_{j=0}^{\infty}\frac{(-1)^j r^j\alpha^j}{j!}\int_0^\infty x^{-r(\lambda+1)}\left(1+x^{-\lambda}\right)^{-(\gamma\beta r+\gamma\beta j+r)}\left(1-\left(1+x^{-\lambda}\right)^{-\gamma}\right)^{-(\beta r+\beta j+r)}dx$$

following the same procedure as in section 2.5, we obtain:

$$\int_0^\infty \left(f(x)\right)^r dx = (\alpha\beta\lambda\gamma)^r \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}(\beta r+\beta j+r)!r^j\alpha^j}{\lambda(\beta r+\beta j+r-k)!k!j!} B(\lambda z - r\lambda - r + 1, r\lambda + r - \lambda) \quad (31)$$

Hence, based on the definition of Rényi entropy, we have:

$$I_R(r) = \frac{1}{1-r}\ell n\left[(\alpha\beta\lambda\gamma)^r \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k}(\beta r+\beta j+r)!r^j\alpha^j}{\lambda(\beta r+\beta j+r-k)!k!j!} B(\lambda z - r\lambda - r + 1, r\lambda + r - \lambda)\right]$$

## 3.9 Maximum Likelihood Estimates of the Parameters of Weibull-Burr III Distribution

Let $x_1, x_2, \cdots, x_n$ denote a random sample drawn from the Weibull-Burr III distribution with parameters $\alpha, \beta, \lambda \; and \; \gamma$ defined in equation (10). The likelihood function, $L(x;\; \alpha, \beta, \lambda, \gamma)$ is defined to be the joint density function of the random variables,$x_1, x_2, \cdots, x_n$. That is,

$$L(x;\alpha,\beta,\lambda,\gamma) = \prod_{i=1}^{n} f(x_i;\alpha,\beta,\lambda,\gamma) \quad (32)$$

Any statistic value that maximizes the likelihood function, $L(x_1, x_2, \cdots, x_n;\; \alpha, \beta, \lambda, \gamma)$, is referred to as the maximum likelihood estimator. The likelihood and log likelihood of the Weibull-Burr III distribution are given by:

$$L(x;\alpha,\beta,\lambda,\gamma) = \prod_{i=1}^{n} \alpha\beta\lambda\gamma x_i^{-(\lambda+1)} \frac{\left(1+x_i^{-\lambda}\right)^{-(\gamma\beta+1)}}{\left(1-\left(1+x_i^{-\lambda}\right)^{-\gamma}\right)^{\beta+1}} \exp\left\{-\alpha\left[\left(1+x_i^{-\lambda}\right)^{\gamma}-1\right]^{-\beta}\right\} \quad (33)$$

and

$$\ell(x;\alpha,\beta,\lambda,\gamma) = n\log\alpha + n\log\beta + n\log\lambda + n\log\gamma - \alpha\sum_{i=1}^{n}\left[\left(1+x_i^{-\lambda}\right)^{\gamma}-1\right]^{-\beta} - (\lambda+1)\sum_{i=1}^{n}\log(x_i) - (\gamma\beta+1)\sum_{i=1}^{n}\log\left(1+x_i^{-\lambda}\right) - (\beta+1)\sum_{i=1}^{n}\left[1-\left(1+x_i^{-\lambda}\right)^{-\gamma}\right] \quad (34)$$

respectively. To obtain the Maximum Likelihood Estimators of the parameters of Weibull-Burr III distribution, we differentiate (34) partially with respect to the parameters and equate to zero. This gives:

$$\frac{\partial\ell(x;\alpha,\beta,\lambda,\gamma)}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\left[\left(1+x_i^{-\lambda}\right)^{\gamma}-1\right]^{-\beta} \quad (35)$$

$$\frac{\partial \ell(x;\alpha,\beta,\lambda,\gamma)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\left[\left(1+x_i^{-\lambda}\right)^{\gamma}-1\right]^{-\beta} \log \sum_{i=1}^{n}\left[\left(1+x_i^{-\lambda}\right)^{\gamma}-1\right] - \gamma\sum_{i=1}^{n}\log\left(1+x_i^{-\lambda}\right) - \sum_{i=1}^{n}\left[1-\left(1+x_i^{-\lambda}\right)^{-\gamma}\right] \tag{36}$$

$$\frac{\partial \ell(x;\alpha,\beta,\lambda,\gamma)}{\partial \lambda} = \frac{n}{\lambda} - \alpha\beta\gamma\sum_{i=1}^{n}\frac{x_i^{-\lambda}\log(x_i)}{\left(1+x_i^{-\lambda}\right)^{1-\gamma}\left[\left(1+x_i^{-\lambda}\right)^{\gamma}-1\right]^{\beta+1}} \tag{37}$$

$$\frac{\partial \ell(x;\alpha,\beta,\lambda,\gamma)}{\partial \gamma} = \frac{n}{\gamma} + \alpha\beta\sum_{i=1}^{n}\frac{\left(1+x_i^{-\lambda}\right)^{\gamma}\log\left(1+x_i^{-\lambda}\right)}{\left[\left(1+x_i^{-\lambda}\right)^{\gamma}-1\right]^{\beta+1}} \tag{38}$$

Solving for $\alpha, \beta, \lambda$ $and$ $\gamma$, we have:

$$\alpha = \frac{n}{\sum_{i=1}^{n}\left[\left(1+x_i^{-\lambda}\right)^{\gamma}-1\right]^{-\beta}} \tag{39}$$

$$\beta = \frac{n}{\gamma\sum_{i=1}^{n}\log\left(1+x_i^{-\lambda}\right) - \sum_{i=1}^{n}\left[\left(1+x_i^{-\lambda}\right)^{\gamma}-1\right]^{-\beta}\log\sum_{i=1}^{n}\left[\left(1+x_i^{-\lambda}\right)^{\gamma}-1\right] + \sum_{i=1}^{n}\left[1-\left(1+x_i^{-\lambda}\right)^{-\gamma}\right]} \tag{40}$$

$$\lambda = \frac{n}{\alpha\beta\gamma\sum_{i=1}^{n}\frac{x_i^{-\lambda}\log(x_i)}{\left(1+x_i^{-\lambda}\right)^{1-\gamma}\left[\left(1+x_i^{-\lambda}\right)^{\gamma}-1\right]^{\beta+1}}} \tag{41}$$

$$\gamma = -\frac{n}{\alpha\beta\sum_{i=1}^{n}\frac{\left(1+x_i^{-\lambda}\right)^{\gamma}\log\left(1+x_i^{-\lambda}\right)}{\left[\left(1+x_i^{-\lambda}\right)^{\gamma}-1\right]^{\beta+1}}} \tag{42}$$

Solving equations (39), (40), (41) and (42) algebraically may be intractable. To avoid this problem, one can obtain the MLEs of $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$ $and$ $\hat{\gamma}$ numerically by applying any of the following methods: Newton–Raphson, Broyden–Fletcher–Goldfarb–Shanno (BFGS), Limited Memory quasi-Newton code for Bound-constrained optimization (L-BFGS-B), Berndt–Hall–Hall–Hausman (BHHH) and Simulated-Annealing (SANN). The Newton-Raphson algorithm uses numerical or analytical gradients and Hessians while BFGS, L-BFGS-B and BHHH algorithms use only numerical or analytical gradients. On the other hand, SANN algorithms use neither gradients nor Hessians but it only uses function values. The Information Matrix of the parameters of the Weibull-Burr III distribution can be derived by differentiating equations (35), (36), (37) and (38) in a situation where we need to obtain interval estimates and test on model parameters.

### 3.10 Data Analysis Procedures

We shall use both the kernel based density estimation technique and the classical Kolmogrov-Smirnov two-sample test procedure to further elucidate on the efficacy of the Weibull-Burr III distribution during the data analysis of real life data. We shall discuss briefly the two techniques here.

#### 3.10.1 The Kernel-based Density Estimation

An estimator of the probability density function $f(x)$ using the kernel approach is given by:

$$f(x) = \frac{1}{n\xi}\sum_{i=1}^{n} k\left(\frac{x - x_i}{\xi}\right), \qquad x > 0 \tag{43}$$

where $k(.)$ is the kernel, $\xi$ is the smoothing parameter and $n$ is the sample size of the real life data. A suitable kernel function, $k(.)$ is the Epanechnikov (1969) kernel. This kernel is defined by:

$$k(r) = \begin{cases} \frac{0.75\left(1 - 0.2r^2\right)}{\sqrt{5}} & if\ |r| \le \sqrt{5} \\ 0 & elsewhere \end{cases} \tag{44}$$

and the parameter, $\xi$ , determines the smoothness of the estimator. Increasing $\xi$ increases the bias but smoothes the estimates. In the spirit of Fiksel (1988), we have used computational and simulation studies to suggest taking the value of $\xi$ as:

$$\xi = \frac{5.5}{\sqrt{\lambda}} \tag{45}$$

where $\lambda$ is the mean of the real life data, as appropriate.

The kernel in equation (44) is symmetric as well as a probability density. This means that the estimator $f(x)$ given in equation (43) will be a probability density function. The estimated Weibull-Burr III, Weibull and Burr III probability density functions will be compared graphically with $f(x)$.

### 3.10.2 Kolmogrov-Simirnov Two-Sample Test

Consider the universe consisting of two populations which are called $X$ and $Y$ populations, with cumulative density functions denoted by $F_X$ and $F_Y$, respectively. We have a random sample of size $m$ from the $X$ population and another random sample of size $n$ drawn independently from population $Y$, i.e

$$X_1, X_2, \cdots, X_m \text{ and } Y_1, Y_2, \cdots, Y_n .$$

Usually, the hypothesis of interest is that the two samples are drawn from identical populations:

$$H_0 : F_Y(x) = F_X(x) \quad \forall x$$

The order statistic corresponding to the two random samples from continuous populations, $F_X$ and $F_Y$, are respectively:

$$X_{(1)}, X_{(2)}, \cdots, X_{(m)} \text{ and } Y_{(1)}, Y_{(2)}, \cdots, Y_{(n)}$$

Their respective empirical distribution functions denoted by $S_m(x)$ and $S_n(x)$ are defined as:

$$S_m(x) = \begin{cases} 0 & \text{if } x < X_{(1)} \\ \frac{k}{m} & \text{if } X_k \le x < X_{k+1} \qquad k = 1, 2, \dots, m-1 \\ 1 & \text{if } x \ge X_{(m)} \end{cases}$$

and

$$S_n(x) = \begin{cases} 0 & \text{if } x < Y_{(1)} \\ \frac{k}{n} & \text{if } Y_k \le x < Y_{k+1} \qquad k = 1, 2, \dots, n-1 \\ 1 & \text{if } x \ge Y_{(n)} \end{cases}$$

In a combined ordered arrangement of the $N = n + m$ sample observations, $S_m(x)$ and $S_n(x)$ are the respective proportions of $X$ and $Y$ observations which do not exceed the specified value, $x$. If the null hypothesis is true, the population distributions are identical. The two-sided Kolmogrov-Smirnov two-sample test criterion, $D_{m,n}$, is based on the maximum absolute difference between the two empirical distributions,

$$D_{m,n} = \underset{x}{Sup}\left|S_m(x) - S_n(x)\right|$$

and the rejection region is the upper tail, defined by:

$$D_{m,n} \geq C_\alpha$$

where

$$P\left(D_{m,n} \geq C_\alpha / H_0\right) = \alpha$$

For m and n large, right-tail critical values based on the asymptotic distribution can be calculated as:

| $\alpha$ | 0.2 | 0.1 | 0.05 | 0.01 |
|---|---|---|---|---|
| $C_\alpha$ | $1.07\sqrt{N/nm}$ | $1.22\sqrt{N/nm}$ | $1.36\sqrt{N/nm}$ | $1.63\sqrt{N/nm}$ |

## 4.0 Data Analysis

Here, an application of Weibull-Burr III distribution is provided by comparing the results of the fits of this model with that of other weibull-G family of distributions. Two data sets will be used in order to make the comparison.

The first data set is the uncensored survival data that was previously analyzed by Lee and Wang (2003), Lemonte and Cordeiro (2011) and Luz *et al*. (2012). This data set will be used to compare between fits of the Weibull-Burr III distribution with that of Weibull, Burr III, Weibull-Exponential, Weibull-Paretor, New Weibull-Paretor and Wiebull-Rayleigh distributions.The data presented in Table 1 (see Appendix), contains the remission times (in months) of a random sample of 128 bladder cancer patients.

**Table 3**: Maximum Likelihood Estimates (MLE), Log-likelihood, AIC, BIC and CAIC for uncensored data fitted to Weibull-G family distributions

| Model | MLE | | Log-likelihood | AIC | BIC | CAIC |
|---|---|---|---|---|---|---|
| | Parameters | Estimates | | | | |
| Weibull | α | 0.09225 | -414.1 | 832.2 | 837.9 | 832.3 |
| | β | 1.05303 | | | | |
| Burr III | λ | 1.03396 | -426.7 | 857.4 | 857.4 | 857.5 |
| | γ | 4.21689 | | | | |
| Weibull-Burr III | α | 1.38023 | -339.7 | 687.5 | 698.9 | 687.8 |
| | β | 6.67205 | | | | |
| | λ | 0.13683 | | | | |
| | γ | 1.30411 | | | | |
| Weibull-Exponential | α | 4.66306 | -419.3 | 844.7 | 853.2 | 844.8 |
| | β | 0.86469 | | | | |
| | λ | 0.01496 | | | | |
| Weibull-Pareto | α | 0.0038 | -411.8 | 829.6 | 838.2 | 829.8 |
| | β | 0.1287 | | | | |
| | λ | 7.7391 | | | | |
| New Weibull-Pareto | α | 0.2288 | -414.1 | 834.2 | 842.7 | 834.4 |
| | β | 1.0479 | | | | |
| | λ | 2.3398 | | | | |
| Weibull-Rayleigh | α | 0.9113 | -505.8 | 1017.6 | 1026.2 | 1017.8 |
| | β | 0.4963 | | | | |
| | λ | 0.0013 | | | | |

The method of maximum likelihood is used to fit the proposed Weibull-Burr III distribution, Weibull distribution, Burr III distribution, Weibull-Exponential distribution, Weibull-Pareto distribution, New Weibull-Pareto distribution and Weibull-Rayleigh distribution to these data. Criteria such as Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Consistent Akaike Information Criterion (CAIC) are used to compare the distribution models.

The distribution model with the smallest AIC, BIC and CAIC values is considered to be the best distribution.

Table 3 shows MLEs of the parameters for each of the fitted distributions and the statistics: AIC, BIC and CAIC. The results from the Weibull-G family of distributions showed that, the proposed Weibull-Burr III distribution has the least AIC, BIC and CAIC values. Hence, this is an indication that Weibull-Burr III distribution is a very strong competitor to other distributions used here for fitting the data set. We have also estimated the unknown probability density function, $f(x)$ of the lifetime data using the kernel density approach. The plots of the $f(x)$ for the lifetime data and the estimated density functions of the same data set using the Weibull-Burr III distribution, Weibull distribution and Burr III distribution are presented in Figure 3. The estimated density function from the proposed distribution closely mimics the estimated kernel based density function than the other two distributions. Also, having estimated the Weibull-Burr III parameters from the uncensored data, we employ the quantile function to generate a sample of size 128 observations. This sample is presented in Table 2.

We now test the null hypothesis that the uncensored remission time data and the sample drawn from the fitted Weibull-Burr III distribution are indeed from the same distribution. The computed Kolmogrov-Smirnov statistic based on the two empirical distribution functions, $S_m(x)$ and $S_n(x)$ (shown in Figure 4) is

$$D_{128,128} = \underset{x}{Sup}\left|S_m(x) - S_n(x)\right|$$

$$= 0.1094.$$

We only reject the null hypothesis that the uncensored remission data and the sample data drawn from the proposed distribution are from the same distribution if $D_{128,128} \geq C_\alpha$ with $P\left(D_{128,128} \geq C_\alpha\right) = 0.01$. Since $D_{128,128} = 0.1094$ is less than the critical value of $C_\alpha = 0.2038$ at the 1% level of significance, this suggests that we have no reason but fail to reject the hypothesis that the two dataset are from the same distribution.

The second data set is the survival times (in months) of a random sample of 101 patients with advanced acute myelogenous leukemia that received either an autologous bone marrow transplant or an allogeneic bone marrow transplant. The data is given in Table 4 (see Appendix).

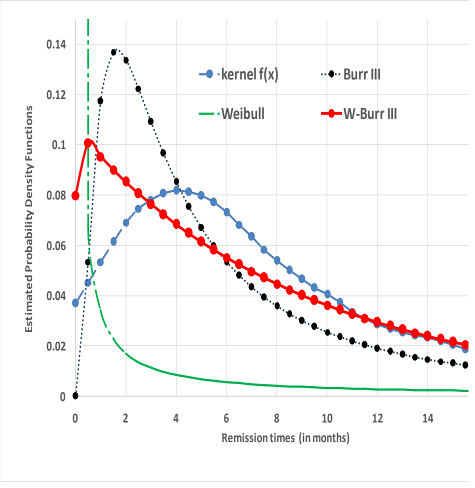


Figure 3: Estimated Probability Density Function for the Remission times data

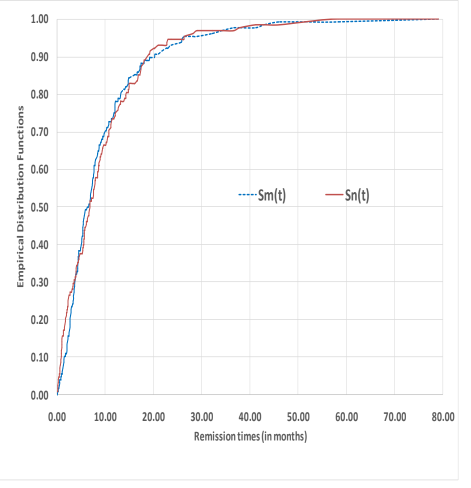


Figure 4: Kolmogrov-Smirnov two sample test: Graph of the empirical distribution Functions

Also in this case, the method of maximum likelihood is used to compare between the fits of the proposed Weibull-Burr III distribution with some of the Weibull-G family distributions. Table 5 shows the MLEs of the parameters for each of the fitted distributions together with the AIC and BIC statistics. This result revealed that the proposed Weibull-Burr III distribution is a very strong competitor compared to other distributions since it is having the least AIC and BIC values.

Table 5: Maximum Likelihood Estimates (MLE), Log-likelihood, AIC and BIC for censored data fitted to Weibull-G family distributions

| **Model** | **MLE** | | **Log-likelihood** | **AIC** | **BIC** |
|---|---|---|---|---|---|
| | **Parameters** | **Estimates** | | | |
| Weibull-Burr III | α | 1.4267 | -219.7 | 447.6 | 458 |
| | β | 0.5705 | | | |
| | λ | 0.4878 | | | |
| | γ | 7.9638 | | | |
| Weibull-Exponential | α | 25.5475 | -880.4 | 1766.9 | 1774.7 |
| | β | 6.8616 | | | |
| | λ | 0.0096 | | | |
| Weibull-Paretor | α | 3.1477 | -223 | 452 | 459.8 |
| | β | 0.1184 | | | |
| | λ | 0.0155 | | | |
| New Weibull -Paretor | α | 0.5112 | -222.4 | 450.9 | 458.7 |
| | β | 0.6752 | | | |
| | λ | 16.0222 | | | |
| Weibull Rayleigh | α | 3.3369 | -599.8 | 1205.7 | 1213.5 |
| | β | 0.0022 | | | |
| | λ | 2.2212 | | | |

## 5.0 Conclusions

In this paper, we proposed a new distribution which generalizes the Burr III distribution. The distribution is named Weibull-Burr III distribution. The pdf, cdf, limit of pdf and hazard function were derived. Additionally, some of the mathematical and statistical properties like moments, skewness, kurtosis, order statistics and entropy were also derived. The model parameters were estimated by using the maximum likelihood estimation procedure. Finally, we fit the proposed model to censored and uncensored data and compared it with estimates from other Weibull-G family distributions. The new distribution was found to provide a better fit than its competitors. The classical Kolmogrov-Simirnov two-sample test statistic, also confirmed that the remission data could have been drawn from the new proposed distribution.

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## Appendix

Table 1: Remission times (in Months) Random Sample of 128 Bladder Cancer Patients

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.08 | 9.22 | 2.62 | 15.96 | 5.49 | 5.85 | 12.07 |
| 2.09 | 13.80 | 3.82 | 36.66 | 7.66 | 8.26 | 21.73 |
| 3.48 | 25.74 | 5.32 | 1.05 | 11.25 | 11.98 | 2.07 |
| 4.87 | 0.50 | 7.32 | 2.69 | 17.14 | 19.13 | 3.36 |
| 6.94 | 2.46 | 10.06 | 4.23 | 79.05 | 1.76 | 6.93 |
| 8.66 | 3.64 | 14.77 | 5.41 | 1.35 | 3.25 | 8.65 |
| 13.11 | 5.09 | 32.15 | 7.62 | 2.87 | 4.50 | 12.63 |
| 23.63 | 7.26 | 2.64 | 10.75 | 5.62 | 6.25 | 22.69 |
| 0.20 | 9.47 | 3.88 | 16.62 | 7.87 | 8.37 | |
| 2.23 | 14.24 | 5.32 | 43.01 | 11.64 | 12.02 | |
| 3.52 | 25.82 | 7.39 | 1.19 | 17.36 | 2.02 | |
| 4.98 | 0.51 | 10.34 | 2.75 | 1.40 | 3.31 | |
| 6.97 | 2.54 | 14.83 | 4.26 | 3.02 | 4.51 | |
| 9.02 | 3.70 | 34.26 | 5.41 | 4.34 | 6.54 | |
| 13.29 | 5.17 | 0.90 | 7.63 | 5.71 | 8.53 | |
| 0.40 | 7.28 | 2.69 | 17.12 | 7.93 | 12.03 | |
| 2.26 | 9.74 | 4.18 | 46.12 | 11.79 | 20.28 | |
| 3.57 | 14.76 | 5.34 | 1.26 | 18.10 | 2.02 | |
| 5.06 | 26.31 | 7.59 | 2.83 | 1.46 | 3.36 | |
| 7.09 | 0.81 | 10.66 | 4.33 | 4.40 | 6.76 | |

Table 2: Random sample of size 128 drawn from Weibull-Distribution: WB ($\alpha$=1.38023, $\beta$=6.67205, $\lambda$=0.13683, $\gamma$=1.30411)

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.00 | 8.91 | 3.89 | 13.99 | 6.50 | 1.30 | 23.02 |
| 0.09 | 9.13 | 3.89 | 14.25 | 6.60 | 1.32 | 26.37 |
| 0.12 | 9.34 | 3.92 | 14.38 | 6.72 | 1.54 | 28.15 |
| 0.20 | 9.40 | 4.04 | 14.76 | 6.76 | 1.61 | 29.01 |
| 0.26 | 9.62 | 4.21 | 14.95 | 6.83 | 1.70 | 37.86 |
| 0.33 | 10.19 | 4.47 | 15.00 | 6.84 | 1.75 | 41.20 |
| 0.45 | 10.44 | 4.53 | 16.57 | 7.12 | 1.88 | 51.51 |
| 0.58 | 10.47 | 4.83 | 16.79 | 7.52 | 1.98 | 58.36 |
| 0.59 | 10.75 | 5.27 | 16.85 | 7.56 | 2.05 | |
| 0.65 | 10.76 | 5.31 | 17.23 | 7.57 | 2.14 | |
| 0.73 | 11.10 | 5.46 | 17.36 | 7.74 | 2.24 | |
| 0.76 | 11.17 | 5.56 | 17.45 | 7.87 | 2.25 | |
| 0.83 | 11.22 | 5.57 | 17.86 | 7.88 | 2.31 | |
| 0.84 | 11.42 | 5.63 | 18.00 | 7.95 | 2.46 | |
| 0.87 | 11.94 | 5.63 | 18.56 | 8.38 | 2.70 | |
| 0.94 | 12.10 | 5.65 | 19.03 | 8.43 | 3.12 | |
| 0.96 | 12.58 | 5.71 | 19.30 | 8.59 | 3.26 | |
| 0.97 | 12.69 | 6.06 | 20.14 | 8.63 | 3.35 | |
| 0.98 | 13.00 | 6.07 | 21.02 | 8.69 | 3.49 | |
| 1.04 | 13.15 | 6.30 | 22.74 | 8.90 | 3.74 | |

Table 4: Survival times (in months) of sample of 101 patients with Advanced Acute Myelogenous leukemia

| | | | | |
|---|---|---|---|---|
| 0.03 | 8.882 | 41.118$^{+}$ | 6.151 | 17.303$^{+}$ |
| 0.493 | 9.145$^{+}$ | 45.033$^{+}$ | 6.217 | 17.664$^{+}$ |
| 0.855 | 11.48 | 46.053$^{+}$ | 6.447$^{+}$ | 18.092 |
| 1.184 | 11.513 | 46.941$^{+}$ | 8.651 | 18.092$^{+}$ |
| 1.283 | 12.105$^{+}$ | 48.289$^{+}$ | 8.717 | 18.75$^{+}$ |
| 1.48 | 12.796 | 57.401$^{+}$ | 9.441$^{+}$ | 20.625$^{+}$ |
| 1.776 | 12.993$^{+}$ | 58.322$^{+}$ | 10.329 | 23.158 |
| 2.138 | 13.849$^{+}$ | 60.625$^{+}$ | 11.48 | 27.73$^{+}$ |
| 2.5 | 16.612$^{+}$ | 0.658 | 12.007 | 31.184$^{+}$ |
| 2.763 | 17.138$^{+}$ | 0.822 | 12.007$^{+}$ | 32.434$^{+}$ |
| 2.993 | 20.066 | 1.414 | 12.237 | 35.921$^{+}$ |
| 3.224 | 20.329$^{+}$ | 2.5 | 12.401$^{+}$ | 42.237$^{+}$ |
| 3.421 | 22.368$^{+}$ | 3.322 | 13.059$^{+}$ | 44.638$^{+}$ |
| 4.178 | 26.776$^{+}$ | 3.816 | 14.474$^{+}$ | 46.48$^{+}$ |
| 4.441$^{+}$ | 28.717$^{+}$ | 4.737 | 15$^{+}$ | 47.467$^{+}$ |
| 5.691 | 28.717$^{+}$ | 4.836$^{+}$ | 15.461 | 48.322$^{+}$ |
| 5.855$^{+}$ | 32.928$^{+}$ | 4.934 | 15.757 | 56.086 |
| 6.941$^{+}$ | 33.783$^{+}$ | 5.033 | 16.48 | |
| 6.941 | 34.211$^{+}$ | 5.757 | 16.711 | |
| 7.993$^{+}$ | 34.77$^{+}$ | 5.855 | 17.204$^{+}$ | |
| 8.882 | 39.539$^{+}$ | 5.987 | 17.237 | |