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## On Numerical Solution for Optimal Allocation of Investment funds in Portfolio Selection Problem

Abubakar Yahaya<sup>1</sup>

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*In this article, we present a procedure for obtaining an optimal solution to the Markowitz's mean-variance portfolio selection problem based on the analytical solution developed in a previous research that lead to the emergence of an important model known as the Black Model. The procedure is well presented, illustrated and validated by a numerical example from real stocks dataset obtainable from a popular European stock market.*

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**Keywords:** Modern Portfolio Theory, Efficient frontier, Pareto optimality, Covariance.

**JEL Classification:** C61, E22, G11.

### 1.0 Introduction

Harry Markowitz's mean-variance portfolio selection model, undoubtedly, serves as the cornerstone upon which the concept of modern portfolio theory (MPT) is founded. The model basically involves selecting some assets from a pool (especially in a stock market) in order to construct a master asset commonly known as portfolio (of assets). The main goal of constructing such portfolio is to 'strike a balance' between mainly two conflicting objectives, namely, making a maximum return/profit at the most minimum risk possible given that a wise choice of constituent assets is made and proper fraction of investment funds are allocated correspondingly.

An apparent common feature of investment opportunities is the fact that their actual returns might significantly vary with their expected values. In a nutshell, we can say they are *risky*. It should be understood at this point that, the concept of financial risk defined by the potential deviation from the expected value is composed of both below and above expected risks outcomes; the latter being as a result of positive surprises or non-occurrence of anticipated negative events. For further details we refer an interested reader to Hallow (1991), Nawrocki (1999), Grootveld and Hallerbach (1999), Ballestero (2005), Estrada (2006), Estrada (2007) and Estrada (2008). On the

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provision that all available information and expectations on future prices are contained in the current prices, then we can regard the future payoffs and returns ( $r_i \mid i = 1, \dots, n$ ) to be treated as random numbers. In simple terms, we can assume that the returns of an asset (say asset  $i$ ), follows a Gaussian distribution in which the expected mean value of the returns,  $\bar{r}_i$ , and their variance,  $\sigma_i^2$ , (or its square root, - which in financial literature is usually referred to as *volatility*) capture all the information concerning the expected outcome, likelihoods and range of deviations from it. See Fama (1970).

According to Yahaya *et al* (2011), when we intend to compare investment opportunities and combine them into portfolios, we need to consider the type and degree of relationship (correlation) existing among their returns. This is to say that when constructing a portfolio of assets an investor has to take into cognizance whether upward/positive deviations in one asset tend to ‘co-move’ with upward/positive or downward/negative deviations in the other assets or even whether there is no interdependence among them. In a situation when assets return are not perfectly positively correlated, then there is a possibility of a scenario in which one asset’s return will be above and another asset’s return below expectation. Hence, positive and negative deviations from the respective expected values will tend to partly offset one another; and consequently the risk involved in combining assets (in a portfolio) is lower than the weighted average of the risks of the individual assets. This intuition has to do with the notion that similar firms (and hence their stocks) perform poorly at the same time, whereas in heterogeneous stocks, some will perform above expectation while others will do worse than expected. Thus, the upward and downward deviations from the expected return will (to some extent) balance, and the actual deviation from the portfolio’s expected return will be smaller than individual asset’s expected return even when both have the same magnitude of expected return.

The rest of this paper is organized in such a way that, the next section gives a brief background of the development and mathematical formulation of the Markowitz model. In section three, we provide a procedure for finding the optimal solution of the model after which a numerical example followed. In section four, we present and discuss the results obtained. Section five provides some concluding remarks.

## 2.0 Markowitz Mean-Variance Portfolio Selection Model

The main goal behind the concept of portfolio management is to combine various securities and other assets into portfolios that address investor needs and then to manage those portfolios so as to achieve the desired investments objectives. The investors' needs are mostly defined in terms of return and risk, and the portfolio manager makes a sound decision aimed at maximizing return for investment risk undertaken. For more details, we refer an interested reader to Yahaya (2004).

The goal of investment decisions which is to maximize shareholders' wealth as well as making sound investment decisions that enhance shareholders' wealth lies at the very heart of the financial manager's job. Wealth enhancing investment decisions (corporate or personal) cannot be made without understanding the interplay between investment returns and investment risk. The risk-return relationship is central to investment decision making, whether evaluating a single investment or choosing between alternative investments. Potential investors, for instance, will assess the risk-return relationship or trade-off in deciding whether to invest in company securities such as shares or bonds. Investors will evaluate whether, in their view, the securities provide return commensurate with their level of risk. For further details, see Yahaya (2004), Etukudo (2010), Di Gaspero *et al* (2011) and Cadenas *et al* (2012).

The classical mean-variance model originally developed by Markowitz is aimed at finding a portfolio of assets that seeks to minimize the risk subject to achieving a given level of return. In this conventional formulation, the portfolio risk (objective function) being minimized is quantified by the portfolio's variance, which is the most commonly used measure. See Markowitz (1952) and Markowitz (1959). The model assumes a market composed of  $n$  assets having corresponding expected returns  $\bar{r}_i$ , and asset covariances  $\sigma_{ij}$ . The aim is to find a set of fractions  $w_i$  of an investor's investment fund to be allotted to each asset  $i$  so as to minimize the risk (variance,  $\sigma_p^2$ ) of the entire portfolio's expected return, while at the same time ensuring that the portfolio's expected return attains a specified target,  $\xi$ . These fractions, or asset weights, must be nonnegative and their sum must be unity. The model can be mathematically formulated as:

$$\text{Minimize} \quad \sigma_p^2 = \sum_{i=1}^n \sum_{j=1}^n w_i \sigma_{ij} w_j \quad 2.1$$

Subject to

$$\sum_{i=1}^n w_i \bar{r}_i = \xi \left( \min_i \bar{r}_i \leq \xi \leq \max_i \bar{r} \right) \quad 2.2$$

$$\sum_{i=1}^n w_i = 1 \quad 2.3$$

$$0 \leq w_i \leq 1 \quad 2.4$$

Equation 2.1 represents the objective function (Portfolio Risk), while equations 2.2 and 2.3 respectively represent the return and budget constraints. Constraint 2.4 ensures that no asset's weight falls outside the interval  $[0, 1]$ , which literally means no short sales are allowed. The above optimization problem can be solved provided the following four conditions hold:

- (i)  $\min_i \bar{r}_i \leq \xi \leq \max_i \bar{r}_i$
- (ii)  $\sigma_i > 0 \quad \forall i$
- (iii)  $-1 < \rho_{ij} < 1 \quad \forall (i, j)$ ; where  $\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$
- (iv)  $\exists (i \neq j)$  such that  $\bar{r}_i \neq \bar{r}_j$

Depending on the covariance matrix  $\{ \text{recall } cov(i, j) = \sigma_{ij} = \rho_{ij} \sigma_i \sigma_j \}$ , the portfolio with the lowest expected return is not necessarily the portfolio with the least risk. In this case the *minimum variance portfolio* has the least risk.

### 3.0 Methodology

#### 3.1 Exact Analytical Solution of Markowitz Model

In this section, we aim to present a procedure developed by Black (1972) for computing optimal portfolios on the efficient frontier when there are no restrictions on the assets' fractions of investment funds, which we will denote by  $w_i$ . Suppose we have  $n$  risky assets. Let the expected return of asset  $i$  be denoted by  $\bar{r}_i$ , the variance of asset  $i$ 's returns by  $\sigma_i^2$ , and the covariance between asset  $i$  and asset  $j$  by  $\sigma_{ij}$ . It should be noted that, since the assets are

risky, the variances are all non-zeros. Let us, furthermore, assume that no asset can be expressed as a linear combination of the other assets, which consequently ensures that the variance-covariance matrix, denoted by  $\mathbf{C} = \{\sigma_{ij} \mid i = 1, 2, \dots, n; j = 1, 2, \dots, n\}$  associated with the  $n$  assets is non-singular, which is a necessary condition for determining its inverse  $\mathbf{C}^{-1}$ .

An efficient portfolio is a feasible portfolio having smallest variance for a given expected return. In other words, it can also be defined as a feasible portfolio having maximum return for a given variance (risk). The efficient frontier associated with these  $n$  assets is a set of efficient portfolios that seem to form a parabolic shape on a risk-return plane after solving the following quadratic optimization problem obtained by dropping the non-negativity constraint provided in equation 2.4 from the classical formulation of the problem provided in section 2 above:

$$\text{Minimize} \quad \sigma_p^2 = \sum_{i=1}^n \sum_{j=1}^n w_i \sigma_{ij} w_j \quad 3.1(a)$$

Subject to

$$\sum_{i=1}^n w_i \bar{r}_i = \xi \quad 3.1(b)$$

$$\sum_{i=1}^n w_i = 1 \quad 3.1(c)$$

Dropping the non-negativity constraints in the above model corresponds to authorizing a practice known as *short selling* – a situation in which the solutions could contain negative assets proportions, and this happens where an investor receives today’s asset price and will have to pay the then current price in future. See Maringer (2005) for more details. Furthermore, it should also be noted that, removing the non-negativity constraints means that any asset’s fraction of investment fund can be any real number (*i.e.*  $w_i \in \mathfrak{R}, \forall i$ ) provided constraint 3.1(c) is satisfied

I guess, what goes in the reader’s mind at this point is that, *why should the non-negativity constraint be dropped in order to determine a solution?* The reason is just that, the inclusion of the non-negativity constraint inhibits the provision of analytic solution and as well transforms the standard Markowitz mean-variance portfolio selection model into a Nondeterministic Polynomial (NP) hard problem. See Garey and Johnson (1979) as well as Arora and Barak (2009).

Now the procedure for determining the optimal solution goes as follows:

Let  $\lambda_1$  and  $\lambda_2$  be Lagrange multipliers, then the Lagrangian formulation of the problem will be:

$$L\{\lambda_1, \lambda_2, (w_i | i = 1, 2, \dots, n)\} = \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij} + \lambda_1 \left( \xi - \sum_{i=1}^n w_i \bar{r}_i \right) + \lambda_2 \left( 1 - \sum_{i=1}^n w_i \right)$$

By treating  $L\{\lambda_1, \lambda_2, (w_i | i = 1, 2, \dots, n)\}$  as a function of  $(n + 2)$  variables  $\{\lambda_1, \lambda_2, (w_i | i = 1, 2, \dots, n)\}$ , the necessary conditions for its *minimum point* are given by:

$$\frac{\partial L}{\partial w_i} \{\lambda_1, \lambda_2, (w_i | i = 1, 2, \dots, n)\} = \sum_{j=1}^n w_j \sigma_{ij} - \lambda_1 \bar{r}_i - \lambda_2 = 0, \quad i = 1, 2, \dots, n \quad 3.1(i)$$

$$\frac{\partial L}{\partial \lambda_1} \{\lambda_1, \lambda_2, (w_i | i = 1, 2, \dots, n)\} = \xi - \sum_{i=1}^n w_i \bar{r}_i = 0 \quad 3.1(ii)$$

$$\frac{\partial L}{\partial \lambda_2} \{\lambda_1, \lambda_2, (w_i | i = 1, 2, \dots, n)\} = 1 - \sum_{i=1}^n w_i = 0 \quad 3.1(iii)$$

Since  $\mathbf{C}$  (the symmetric variance-covariance matrix) was assumed to be non-singular, the asset's weights  $(w_i | i = 1, 2, \dots, n)$  that respect the provision of the above conditions should minimize the portfolio variance  $\sigma_p^2$ .

It can easily be seen that, equation 3.1(i) above defines a linear system of  $n$  equations, where the  $n$  unknowns are  $(w_i | i = 1, 2, \dots, n)$ . Suppose now we denote the elements of the inverse matrix of  $\mathbf{C}$ ,  $\mathbf{C}^{-1}$  by  $c_{ij}$  [*i.e.*  $\mathbf{C}^{-1} = \{c_{ij} | i, j = 1, 2, \dots, n\}$ ], then the system solutions are given by:

$$w_k = \lambda_1 \sum_{j=1}^n c_{kj} \bar{r}_j + \lambda_2 \sum_{j=1}^n c_{kj}, \quad k = 1, \dots, n \quad 3.1(iv)$$

On one hand, in order to utilize equations 3.1(ii) and 3.1(iii), we multiply 3.1(iv) by  $\bar{r}_k$  and summing over from  $k = 1$  to  $n$  to have:

$$\sum_{k=1}^n w_k \bar{r}_k = \lambda_1 \sum_{k=1}^n \sum_{j=1}^n c_{kj} \bar{r}_j \bar{r}_k + \lambda_2 \sum_{k=1}^n \sum_{j=1}^n c_{kj} \bar{r}_k \tag{3.1(v)}$$

While on the other hand we sum over equation 3.1(iv) from  $k = 1$  to  $n$  to have:

$$\sum_{k=1}^n w_k = \lambda_1 \sum_{k=1}^n \sum_{j=1}^n c_{kj} \bar{r}_j + \lambda_2 \sum_{k=1}^n \sum_{j=1}^n c_{kj} \tag{3.1(vi)}$$

Suppose we now define the following constants:

$$\alpha = \sum_{k=1}^n \sum_{j=1}^n c_{kj} \bar{r}_j \tag{3.1(vii)}$$

$$\beta = \sum_{k=1}^n \sum_{j=1}^n c_{kj} \bar{r}_j \bar{r}_k \tag{3.1(viii)}$$

$$\gamma = \sum_{k=1}^n \sum_{j=1}^n c_{kj} \tag{3.1(ix)}$$

Using equations 3.1(v), 3.1(vii) and 3.1(viii); we can now rewrite equation 3.1(ii) as:

$$\xi = \beta \lambda_1 + \alpha \lambda_2 \tag{3.1(x)}$$

In similar fashion, if we use equations 3.1(vi), 3.1(vii) and 3.1(ix); we can also rewrite 3.1(iii) as:

$$1 = \alpha \lambda_1 + \gamma \lambda_2 \tag{3.1(xi)}$$

From equations 3.1(x) and 3.1(xi), we obtain the following system of linear equations:

$$\xi = \beta \lambda_1 + \alpha \lambda_2$$

$$1 = \alpha \lambda_1 + \gamma \lambda_2$$

By solving the above equations simultaneously, the Lagrange multipliers  $\lambda_1$  and  $\lambda_2$  respectively take on values:

$$\lambda_1 = \frac{\gamma \xi - \alpha}{\beta \gamma - \alpha^2} \tag{3.1(xii)}$$

$$\lambda_2 = \frac{\beta - \alpha \xi}{\beta \gamma - \alpha^2} \tag{3.1(xiii)}$$



If we substitute 3.1(xii) and 3.1(xiii) for values of  $\lambda_1$  and  $\lambda_2$  respectively into 3.1(iv), we obtain:

$$w_k^* = \frac{\xi \sum_{j=1}^n c_{kj} (\gamma \bar{r}_j - \alpha) + \sum_{j=1}^n c_{kj} (\beta - \alpha \bar{r}_j)}{\beta \gamma - \alpha^2}, \quad k = 1, \dots, n \quad 3.1(xiv)$$

The solution obtained in equation 3.1(xiv) above completely characterizes the composition of the smallest variance (optimal) portfolio for a given target return,  $\xi$ .

### 3.2 Numerical Illustration

In order to numerically illustrate how the above formulae can be utilized to obtain an optimal solution of a certain portfolio given a specific target return, we downloaded some weekly historical stocks prices of three well-known companies, namely: Amlin Plc (AML), British Sky Broadcasting Group PLC (BSY) and British Petroleum (BP) from FTSE 100 index traded at London Stocks Exchange. The data comprise of 20-weeks stock prices from 18<sup>th</sup> August, 2008 to 29<sup>th</sup> December, 2008.

**Table 1:** Stocks weekly price data obtained from Yahoo Finance site

Week	(AML)	(BSY)	(BP)	Week	(AML)	(BSY)	(BP)
1	288.25	449.75	519.25	11	316.75	378.50	507.25
2	290.75	466.00	528.75	12	330.75	420.25	515.00
3	277.00	445.25	499.25	13	368.00	403.00	488.00
4	278.50	455.50	510.00	14	339.50	370.75	462.25
5	307.25	443.50	490.25	15	366.25	439.50	526.75
6	305.75	426.25	488.50	16	375.00	420.50	478.00
7	315.75	435.50	467.75	17	370.00	465.00	516.25
8	291.25	380.50	376.25	18	370.00	483.25	505.00
9	287.75	368.00	431.75	19	358.00	468.25	496.00
10	270.50	329.00	440.00	20	350.75	482.00	552.75

We should remember that, the main goal of this numerical example is to show how to obtain an optimal allocation of the investment funds to the three stocks that makes up our portfolio, given a desired level of portfolio return. This optimal allocation should be one such that, the specified target return is achieved at the most minimum value of portfolio risk. Let's now begin by denoting company 1 (AML) as stock 1, BSY as stock 2 and BP as stock 3. We

then use the following formula to compute the corresponding stocks weekly

returns: 
$$r_{i,t+1} = 100 \times \ln \left( \frac{P_{i,t+1}}{P_{i,t}} \right); i = 1, 2, 3 \text{ and } t = 0, \dots, 19$$

Where  $i$  denotes the stock number,  $t$  denotes the time period in weeks and  $P_{i,t}$  denotes stock  $i$ 's price at week  $t$ . We now present a table showing the computed weekly returns together with their averages and standard deviations as follows (rounded to 10 decimal places):

**Table 2:** Weekly returns data obtained from TABLE 1 above

$t$	Stock 1	Stock 2	Stock 3	$t$	Stock 1	Stock 2	Stock 3
0	-	-	-	10	15.7840720915	14.0158322112	14.2229251794
1	0.8635632250	3.5493761295	1.8130267604	11	4.3249983794	10.4633699411	1.5162921958
2	-4.8446285211	-4.5549711956	-5.7408758065	12	10.6720135174	-4.1913210342	-5.3851494811
3	0.5400553180	2.2759794530	2.1303753422	13	-8.0608991170	-8.3408580929	-5.4209535586
4	9.8243689079	-2.6697914950	-3.9495260642	14	7.5842213333	17.0109736107	13.0620182417
5	-0.4893973879	-3.9671773152	-0.3575993594	15	2.3609865639	-4.4193237712	-9.7115320193
6	3.2182986663	2.1468767695	-4.3405506329	16	-1.3423020332	10.0592926174	7.6980411784
7	-8.0768756351	-13.5008618991	-21.7680149091	17	0.0000000000	3.8496712501	-2.2032715000
8	-1.2089957278	-3.3403239133	13.7592900964	18	-3.2970019238	-3.1531776764	-1.7982502550
9	-6.1819949315	-11.2025187404	1.8928009886	19	-2.0459267418	2.8941772726	10.8329893013
$\bar{r}_i$	Average returns,				1.032871368	0.36448548	0.32905451
$\sigma_i$	Standard Deviation,				6.424488785	8.228335594	8.983470314

We now use Table 2 above, to determine the variance-covariance matrix,  $\mathbf{C} = \{\sigma_{ij} \mid i, j = 1, 2, 3\}$  and the inverse matrix of  $\mathbf{C}$ , denoted as  $\mathbf{C}^{-1} = \{c_{ij} \mid i, j = 1, 2, 3\}$ . Therefore, the actual values of  $\mathbf{C}$  and  $\mathbf{C}^{-1}$  are as given below:

$$\mathbf{C} = \{\sigma_{ij} \mid i, j = 1, 2, 3\} = \begin{pmatrix} 41.2740561450309 & 30.6687682398956 & 20.3055942449002 \\ 30.6687682398956 & 67.7055066400574 & 48.3480585386871 \\ 20.3055942449002 & 48.3480585386871 & 80.7027388743473 \end{pmatrix}$$

$$\mathbf{C}^{-1} = \{c_{ij} \mid i, j = 1, 2, 3\} = \begin{pmatrix} 0.0365940390041912 & -0.0174785802991764 & 0.00126379497275011 \\ -0.0174785802991764 & 0.0341609850496135 & -0.0160676621818857 \\ 0.00126379497275011 & -0.0160676621818857 & 0.0216991168856215 \end{pmatrix}$$

Now by using the inverse matrix  $\mathbf{C}^{-1}$  provided above, the corresponding values of  $\alpha$ ,  $\beta$  and  $\gamma$  obtained using equations 3.1(vii), 3.1(viii) and 3.1(ix) are 0.0235421253624985, 0.0297718413449647 and 0.0278892459228022 respectively. Now suppose we want to construct a portfolio with a target return of 84.5% (i.e.  $\zeta = 0.845$ ); then by applying the formula provided in equation 3.1(xiv); we found out that, the optimal solution is given by:

$$\mathbf{w}^* = (w_i^* | i = 1, 2, 3) = \begin{pmatrix} 0.732008707737021 \\ 0.021038699126778 \\ 0.246952593136201 \end{pmatrix} \quad 3.2(i)$$

By substituting the values provided above in equations 3.1(a) and 3.1(b), the corresponding portfolio risk and return will be:

$$\begin{pmatrix} \sigma_P^2 \\ \xi \\ \zeta \end{pmatrix} = \begin{pmatrix} 35.8561912851607 \\ 0.8450 \end{pmatrix} \quad 3.2(ii)$$

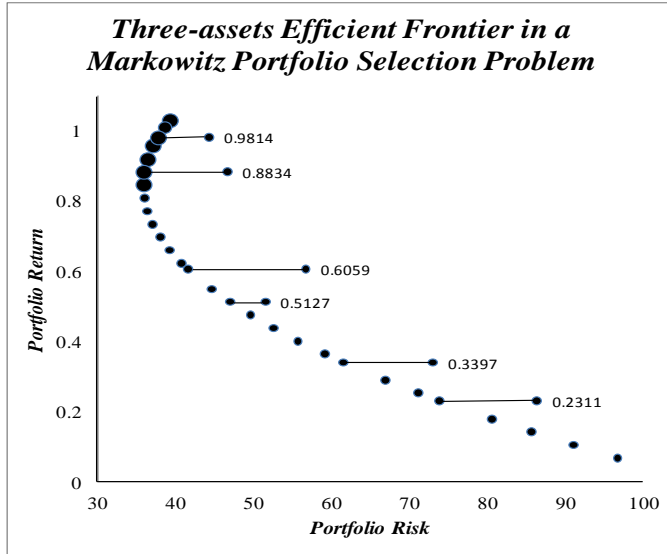
In order to show that the above solution is indeed optimal, we have to show that a slight deviation from the optimal solution will result in a portfolio with a higher risk even if the resultant portfolio managed to produce the same desired (target) return as the optimal. Now for the sake of illustration, let our arbitrary (slightly-deviated optimal) solution be:

$$\mathbf{w} = (w_i | i = 1, 2, 3) = \begin{pmatrix} 0.73201 \\ 0.02104 \\ 0.24695 \end{pmatrix} \quad 3.2(iii)$$

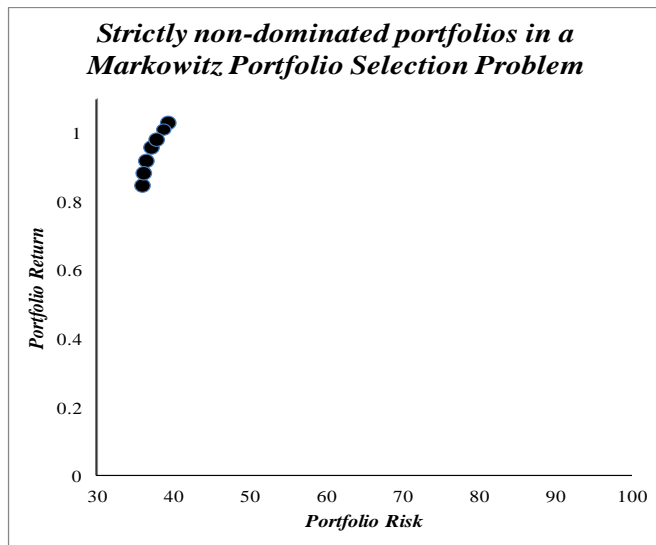
Now by substituting the arbitrary solution above in equations 3.1(a) and 3.1(b), we found out that, although, the target return of 84.5% (i.e.  $\xi = 0.845$ ) has been achieved (as in the optimal solution); however, it is interesting to note that there is a very slight and negligible increase in the value of the portfolio risk. This is because the corresponding portfolio risk and return is now:

$$\begin{pmatrix} \sigma_P^2 \\ \xi \\ \zeta \end{pmatrix} = \begin{pmatrix} 35.856191\mathbf{4536605} \\ 0.8450 \end{pmatrix} \quad 3.2(iv)$$

The following figures shows some set of portfolios (both optimal and sub-optimal) obtained by investing varying proportions of investment funds to the three assets that makes up a given portfolio having a desired target return.



**Figure 1:** Efficient Frontier of Portfolios



**Figure 2:** Strictly non-dominated Portfolios

#### 4.0 Results and Discussions

In the previous section we provided a thorough and a fully detailed procedure for making a sound and intelligent allocation of investment funds to the assets that makes up a given portfolio. It has also been shown in a given numerical example, that the solution provided by the formula in equation 3.1(xiv), indeed provides an *optimum* solution. This is because if we consider the example given, we find out that, when we supplied a desired target portfolio return of

0.8450, the formula in equation 3.1(xiv) provided us with a numerical solution provided in equation 3.2(i) resulting in a portfolio risk and return provided in equation 3.1(ii).

Now in order to check if the so-called optimal solution is indeed optimal, we decided to (very slightly) perturb the solution as provided in equation 3.1(iii) which results in portfolio risk and return provided in equation 3.1(iv). Now by taking a proper look at equations 3.1(ii) and 3.1(iv), we find out that, although there is no difference in both target returns, the portfolio risk of the sub-optimal solution is seen to be slightly higher than the optimal one by a very negligible value of 0.0000001685 ( $35.8561914536605 - 35.8561912851607$ ). If we now take a proper look at Figure 1 above, we can see that, all (efficient) minimum-risk portfolios are seen to make a parabolic-shape frontier of points (portfolios) known as efficient frontier. It can also be seen that, all efficient portfolios do really have minimum-risk than all the non-efficient ones. One vital feature of the efficient frontier as can be seen on Figure 1 also, is that all portfolios (points) can either fall on the efficient frontier (if the portfolios are efficient) or fall on the right hand side of the frontier (if the portfolios are non-efficient), but never on the left-hand side of the efficient frontier. For instance, it can easily be observed (from Figure I) that all the *labeled* points (0.9814, 0.8834, 0.6059, 0.5127, 0.3397, and 0.2311) constitute a set of non-efficient portfolios; this is so, because for any of those portfolios (points) there is a portfolio on the frontier that offers the same magnitude of portfolio return but at lower risk value. Hence non-efficient portfolios are always dominated by the portfolios on the efficient frontier.

Furthermore, from Figure 2 – which results directly from Figure 1 by removing all dominated portfolios, we can see a portion of the efficient frontier that makes up of a set of what (in financial literature) is regarded as strictly *non-dominated* portfolios. These portfolios provide “better value” to the investor than any non-efficient/dominated portfolio chosen by him/her. These portfolios form a set from which a risk-averse (risk-hating) investor mostly makes his/her choices from depending on his/her degree of risk averseness, knowing that if he/she goes vertically there is always a portfolio that offers more return at higher risk. This concept is known in optimization literature as Pareto optimality, which implies that an investor cannot improve one objective without making the other worse. For instance, if an investor

wants to reduce portfolio risk, he should be ready to accept a lower return. One the other hand, if he/she wants to have more return, he/she should be ready to accept more risk.

## 5.0 Conclusion

Based on the analytical and numerical solutions provided in section 3 above, it can easily be understood that, there is an efficient tool within the reach of a Markowitz mean-variance investor to make an intelligent decision of allocating investment funds to the assets that make up the portfolio. We also learnt that, the portfolios on the efficient frontier are always non-dominated in the sense that, any portfolio on efficient frontier offers a better return than another (off-efficient frontier) portfolio having the same degree of portfolio risk. Non-domination in this sense, may also mean, the ability of an efficient portfolio to provide the same level of portfolio return but at lower risk.

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